Noncausal Itô formula and its applications

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(i) Subject and materials: Given smooth functions $a(t,x), b(t,x), a(x), b(x) \in C_b^2$ we consider the Cauchy problems of the noncausal SDEs

(Eq-0) $dX_t = a(t,X_t)d_*W_t + b(t,X_t)dt, \ t \geq 0, \quad X_0 = x_0$

where $W_t(\omega)$ is a real Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and the symbol $\int d_*W_t$ stands for the **noncausal stochastic integral**([2], [6]) with respect to a *regular basis* in $L^2([0,T], dt)$,(see, [6], [1]) and the initial data x_0 is a real number or *can be a random variable* $x_0(\omega)$.

The problem (Eq-0) was first studied by the author ([3]) and the basic properties, such as the existence and uniqueness of solutions, have been established in the cited paper (see also [1]).

Theorem A([3],[1]), *(1), There exists such a solution, say X_t, of the Cauchy problem which satisfies for any $G(t,x) \in \mathbf{C}_b^2$ the following equality (NIF) called* the **noncausal Itô formula**:

$$\text{(NIF)} \quad \begin{aligned} dG(t,X_t) &= G'_t(t,X_t)dt + G'_x(t,X_t)dX_t \\ &= G'_x(t,X_t)\{a(t,X_t)d_*W_t + b(t,X_t)dt\} + G'_t(t,X_t)dt. \end{aligned}$$

We call such solution X_t, the **regular solution** *of the (Eq-0).*
(2) The regular solution is unique **when** $|a(t,x)| > 0$.

Theorem B([5],[1]) *The problem (Eq 0) with random initial data has a regular solution which is unique when* $|a(t,x)| > 0$.

(ii) Aim of the talk: The Itô formula plays the essential rôle in the theory and applications of Itô calculus. So does the *Nonausal Itô formula* in the theory (ie. causl and noncausal) of stochastic calculus. The aim of the talk is to show its power by pickking up, from recent papers ([4],[5],[1]), two typical examples of application.

(Q1) To establish *Mean value theorem* for stochastic integrals: Given a function $f(x)$ and $[s,t]$ and a regular solution X_t of the SDE (Eq-1),

(Eq-1) $dX_t = a(X_t)d_*W_t + b(X_t)dt, \ t \geq 0, \quad X_0 = x_0,$

We will show that for these stochastic ifntegrals, $\int_s^t f(X_r)dX_r$, $\int_s^t f(X_r)d_*W_r$, $\int_s^t f(X_r)d_0W_r$, several mean value formulae can be established, for instance.

Theorem 1([4]) $^\exists\theta_1, ^\exists\theta_2 \in (0,1)$ *such that*

$$\int_s^t f(X_r)d_*W_r = G(I(\theta_1, X, X_t))) \cdot (X_t - X_s) - H(X(I(\theta_2, s, t))) \cdot (t-s),$$

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where

$$G(x) = \frac{f(x)}{a(x)},\ H(x) = f(x)\frac{b(x)}{a(x)},\quad I(\theta, \alpha, \beta) = \alpha + \theta(\beta - \alpha).$$

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(Q2) The verification of the Wong-Zakai's thoerem: Given an approximate sequence $\{W_n(t)\}$, such that $W_n(t) \to W(t)$ unif in $t \in [0,T]$ as $n \to \infty$, and a sequqnce of functions $\{X_n(t)\}$ determined by $dX_n(t) = a(t, X_n)dW_n(t) + b(t, X_n)dt,\ X_n(0) = x_0$.

We show the following results:

Theorem 2*([5])* Let X_t be the regular solution of (Eq-0), then it holds that
$P[\lim_{n\to\infty} \sup_t |X_t - X_n(t)| = 0] = 1.$

Moreover we will show noncausal extensions of these results, Theorem 1 and 2, as we see below:

Theorem 3 ([4],[5]) *Theorem 1, Theorem 2 presented in (Q1) and (Q2) can be extended to the case where X_t are replaced by regular solutions of Cuchy problem of genuine noncausal SDE (Eq-1) and (Eq-0) respectively. with a random initial data $x_0(\omega)$. ([4],[5]),*

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